



PHYS 101 – General Physics I Final Exam

Duration: 120 minutes

Thursday, 26 December 2024; 12:30

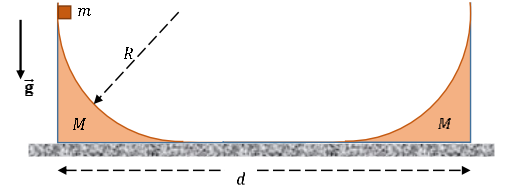
1. Two identical blocks, each with a mass of M , are initially at rest and can slide on a flat, smooth horizontal surface with no friction. The sides of the blocks that face each other are shaped like a quarter circle with a radius of R . A small object with a mass of m , also initially at rest, starts to slide down from the top of the quarter circle on the left block's curved side. There is no friction between m and M .

(a) (7 Pts.) What will be the velocity of the mass m when it reaches the horizontal surface?

(b) (6 Pts.) What distance does mass M move as the mass m moves from the top to the horizontal surface?

(c) (6 Pts.) The small mass continues to slide on the horizontal surface and eventually collides with the block on the right. What is the velocity of the mass when it is at rest relative to the block with which it collides?

(d) (6 Pts.) How high up the block does it rise before coming to rest relative to the block?



Solution: (a) Since there is no net external force in the horizontal direction, the horizontal component of the linear momentum is conserved. Let v_x , V_x denote horizontal velocities of m and M , respectively.

$$p_{xi} = 0, \quad p_{xf} = mv_{xf} + MV_{xf} \rightarrow V_{xf} = \frac{m}{M} v_{xf}$$

Since there is no dissipative force, mechanical energy of the system is also conserved.

$$E_i = mgR, \quad E_f = \frac{1}{2}mv_{xf}^2 + \frac{1}{2}MV_{xf}^2 \rightarrow mgR = \frac{1}{2}mv_{xf}^2 + \frac{1}{2}M\left(\frac{m}{M}v_{xf}\right)^2 \rightarrow v_{xf} = \sqrt{\frac{2MgR}{M+m}}$$

(b) Let Δx and ΔX denote displacements of m and M during the motion. Since the center of mass of the system does not move during the motion, we have

$$m \Delta x + M \Delta X = 0 \rightarrow \Delta x = -\frac{M}{m} \Delta X$$

The displacement of the mass m relative to M is equal to R , meaning that $\Delta X - \Delta x = R \rightarrow \Delta X = \frac{mR}{M+m}$.

(c) In this case the velocities of the two blocks are equal, and the horizontal component of the linear momentum is conserved. Let v' denote the common velocity of the two masses when m is at rest relative to M . We have

$$m \sqrt{\frac{2MgR}{M+m}} = (M+m) v' \rightarrow v' = \sqrt{\frac{2m^2MgR}{(M+m)^3}}$$

(d) Since mechanical energy is also conserved, letting h denote the maximum height m reaches, we find

$$E_i = \frac{1}{2}m \left(\frac{2MgR}{M+m} \right), \quad E_f = \frac{1}{2}(M+m) \left(\frac{2m^2MgR}{(M+m)^3} \right) + mgh \rightarrow h = \left(\frac{M}{M+m} \right)^2 R$$

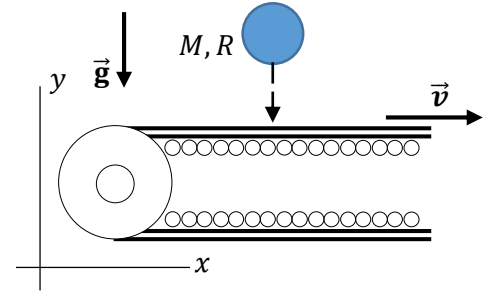
2. A cylinder with mass M and radius R ($I = \frac{1}{2}MR^2$), initially at rest, is softly placed on a horizontal conveyor belt. The speed of the conveyor belt is v , and the coefficient of kinetic friction between the cylinder and the belt is μ_k . The cylinder will initially slide until it starts to roll without slipping on the belt.

(a) (4 Pts.) Draw a free body diagram of the cylinder.

(b) (7 Pts.) What is the velocity (magnitude and direction) of the center of the cylinder relative to the ground when it starts to roll?

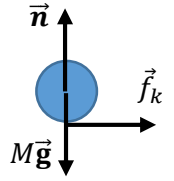
(c) (7 Pts.) When it is rolling, what is the angular speed of the cylinder? Is it rotating clockwise or counter clockwise?

(d) (7 Pts.) Find the work done by the friction force on the cylinder.



Solution: (a) Free body diagram of the cylinder when it is placed on the conveyor belt is as shown.

$$f_k = \mu_k Mg \rightarrow f_k = Ma_c \rightarrow a_c = \mu_k g$$



(b) Torque on the cylinder is counter clockwise with magnitude $\tau = Rf_k = \mu_k MgR$. The angular acceleration of the cylinder is

$$\tau = I\alpha \rightarrow \alpha = \frac{\mu_k MgR}{I} = \frac{2\mu_k g}{R}$$

Therefore, velocity of the center of the cylinder and its angular speed at any time is given by

$$v_c = a_c t = \mu_k g t, \quad \omega = \alpha t = \left(\frac{2\mu_k g}{R}\right)t$$

The cylinder will start to roll when its velocity relative to the conveyor belt satisfies $v_{cr} = v_c - v = -R\omega$, meaning that

$$\mu_k g t_R - v = -R\left(\frac{2\mu_k g}{R}\right)t_R \rightarrow t_R = \frac{v}{3\mu_k g}$$

So, the velocity of the center of the cylinder is to the right $\vec{v}_c = a_c t_R \hat{i} = (v/3)\hat{i}$.

Its angular speed is

$$\omega(t_R) = \left(\frac{2\mu_k g}{R}\right)\left(\frac{v}{3\mu_k g}\right) \rightarrow \omega(t_R) = \frac{2v}{3R}$$

and is counter clockwise.

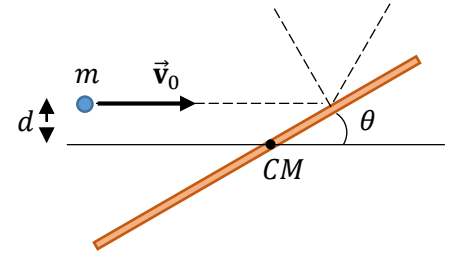
(c) Friction force does positive work on the cylinder as the cylinder accelerates. Initial and final kinetic energies of the cylinder is

$$K_i = 0, \quad K_f = \frac{1}{2}Mv_{\text{rot}}^2 + \frac{1}{2}I\omega_{\text{rot}}^2 = \frac{1}{2}M\left(\frac{v}{3}\right)^2 + \frac{1}{2}\left(\frac{MR^2}{2}\right)\left(\frac{2v}{3R}\right)^2 = \frac{1}{6}Mv^2$$

Since friction is the only horizontal force moving the cylinder, according to the work-energy theorem

$$W_f = K_f - K_i \rightarrow W_f = \frac{1}{6}Mv^2.$$

3. A thin rod of moment of inertia I_{CM} around its center of mass is at rest on a frictionless horizontal surface. The center of mass of the rod is fixed, and the rod is free to rotate about its center of mass without friction. An object (puck) sliding on the horizontal surface with velocity $\vec{v}_0$ collides with the rod as shown in the figure.



(a) (10 Pts.) The object sticks to the rod after the collision, and the rod and the object start to rotate about the fixed point. What is the angular speed?

(b) (15 Pts.) Suppose that instead of sticking to it, the object bounces off the rod after an elastic collision. During this bounce, the angle between its initial direction of motion and the perpendicular line to the rod is the same as the angle between its final direction of motion and the perpendicular line. Find the velocity of the object after the collision. What is the angular speed of the rod in this case? Is such an outcome always possible? Explain.

Solution: (a) The object sticks to the rod at a distance $d/\sin \theta$ from the center of mass of the rod. The moment of inertia of the rod plus the object around the center of mass of the rod is $I = I_{CM} + md^2/\sin^2 \theta$. Angular momentum of the system around the center of mass of the rod is conserved, therefore

$$L_i = mv_0 d, \quad L_f = I\omega \rightarrow \omega = \frac{mv_0 d}{I_{CM} + md^2/\sin^2 \theta}$$

(b) In this case both angular momentum and kinetic energy is conserved. From the geometry of the figure, we see that the angular momentum of the object with respect to the center of the rod after the collision is $L_f = -mvd$, where v is the speed of the object after the collision. Angular momentum conservation gives

$$mv_0 d = -mvd + I_{CM}\omega \rightarrow v = \left(\frac{I_{CM}}{md}\right)\omega - v_0$$

where $I_{CM} = ML^2/12$ is the moment of inertia of the rod around its center of mass, and ω is its angular speed after the collision. Since kinetic energy is also conserved, we have

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}I_{CM}\omega^2$$

Using the result of the first equation in the second one, we find the equation

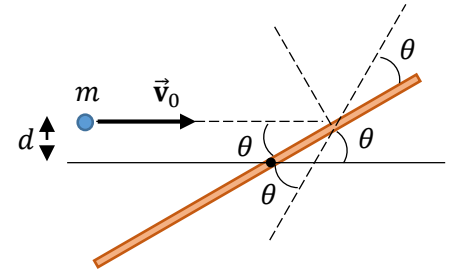
$$\left(\frac{I_{CM}}{md} + d\right)\omega^2 - 2v_0\omega = 0 \rightarrow \omega = 0 \text{ or } \omega = \frac{2mv_0 d}{I_{CM} + md^2}$$

The root $\omega = 0$ corresponds to the situation before the collision.

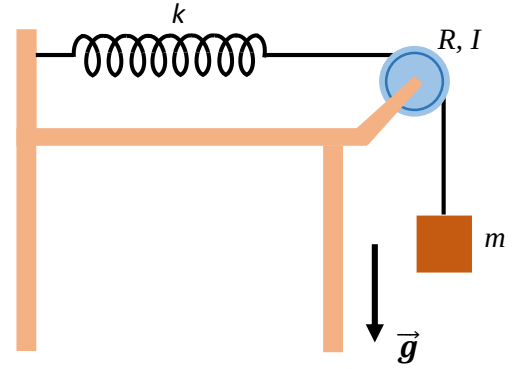
The speed of the object after the collision is

$$v = \left(\frac{I_{CM}}{md}\right)\left(\frac{2mv_0 d}{I_{CM} + md^2}\right) - v_0 \rightarrow v = \left(\frac{I_{CM} - md^2}{I_{CM} + md^2}\right)v_0$$

Since the speed must be positive, we see that such a situation is possible if $I_{CM} > md^2$.



4. Consider the system illustrated in the figure. The pulley with radius R and moment of inertia I around its fixed axis is mounted on a frictionless axle which is fixed to the table. The spring with stiffness constant k has one end fixed to the table, and the other end is attached to a massless inextensible rope. The rope passes over a pulley, and a mass m hangs at its other end. Initially, the system is at rest.



(a) (5 Pts.) How much is the spring stretched from its relaxed (natural) length when the system is at rest in equilibrium?

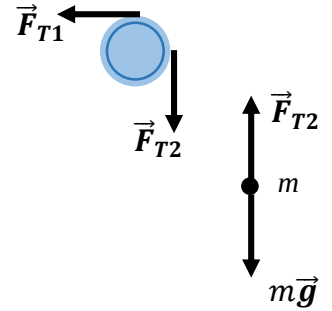
(b) (20 Pts.) If the hanging mass is slightly displaced downward and then released, the system will oscillate around the equilibrium configuration. Find the frequency of small oscillations. Assume that the rope does not slide over the pulley, the pulley can rotate without friction, and the rope between the pulley and the hanging mass remains vertical.

Solution:

(a) In the equilibrium configuration the system is at rest. Hence, we have

$$kx_0 = F_{T1} = F_{T2} = mg \rightarrow x_0 = \frac{mg}{k},$$

where x_0 is the amount the spring is stretched.



(b) If the spring is further stretched by a small amount x , we have

$$F_{T1} = k(x + x_0), \quad RF_{T1} - RF_{T2} = I\alpha, \quad F_{T2} - mg = ma.$$

If the rope does not slide over the pulley, we have $a = R\alpha$. The force is a restoring force in the direction of decreasing x , meaning that

$$a = -\frac{d^2x}{dt^2}.$$

Using these, and eliminating the tension forces in the equations, we obtain

$$kx + kx_0 - mg = -\left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2}.$$

Using the equilibrium condition $x_0 = mg/k$, we have the equation

$$\frac{d^2x}{dt^2} + \frac{k}{\left(m + \frac{I}{R^2}\right)} x = 0$$

Hence we see that $\omega^2 = k\left(m + \frac{I}{R^2}\right)^{-1}$, and therefore

$$\omega^2 = \frac{k}{\left(m + \frac{I}{R^2}\right)} \rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{\left(m + \frac{I}{R^2}\right)}}.$$